Spatial Continuum Model: Toward the Fundamental Limits of Dense Wireless Networks

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***Abstract*—This paper proposes a new model called *spatial continuum asymmetric channels* to study the channel capacity region of asymmetric scenarios in which either one source transmits to a spatial density of receivers or a density of transmitters transmit to a unique receiver. This approach is built upon the classical broadcast channel (BC) and multiple access channel (MAC). For the sake of consistency, the study is limited to Gaussian channels with power constraints and is restricted to the asymptotic regime (zero-error capacity).**

The reference scenario comprises one base station in Tx or Rx mode, a spatial random distribution of nodes (resp. in Rx or Tx mode) characterized by a probability spatial density of users $u(x)$ where each of them requests a quantity of information with no delay constraint, thus leading to a requested rate spatial density $\rho(x)$. This system is modeled as an $\infty-$user asymmetric channel (BC or MAC). To derive the fundamental limits of this model, a spatial discretization is first proposed to obtain an equivalent BC or MAC. Then, a specific sequence of discretized spaces is defined to refine infinitely the approximation. Achievability and capacity results are obtained in the limit of this sequence while the access capacity region $\mathcal{D}_\Omega(P_m)$ is defined as the set of requested rates spatial densities $\rho(x)$ that are achievable with a transmission power P_m. The uniform capacity defined as the maximal symmetric achievable rate is also computed.

I. Introduction

One of the most challenging objectives of 5G is to connect the billions of objects related to the Internet of Things (IoT). Importantly, IoT is made of a very dense set of communicating *things* (called hereafter the nodes) but with low transmission probabilities. An important problem that arises in this paradigm is to determine the maximal density of wireless nodes a base station (BS) may support under some quality of service (QoS) constraints, e.g. packet size, error probability, etc. The answer relies on determining the set of spatial densities of data rate requests that are achievable under some power constraint. Such a set is equivalent to a Shannon capacity region for a cellular system and its bound would represent the fundamental energy efficiency spectral efficiency (EE-SE) limit.

To address this issue, this paper proposes a new continnum based approach. Simplified scenarios are studied but they pave the way for future more complex system analyses. *Asymmetric spatial continuum channels* refer to scenarios in which either a unique transmitter sends information to a spatial density of receivers (downlink mode, represented by a spatial continuum broadcast channel, abbreviated SCBC), or in which a spatial density of transmitters send independent information streams to a unique receiver (uplink mode represented by a spatial continuum multiple access channel, abbreviated SCMAC). The unique transmitter (resp. receiver) is the BS. The distributed nodes are characterized with a spatial density function $u(x)$.

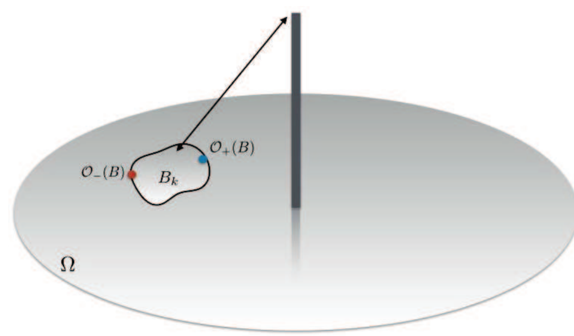

Fig. 1. A PF node (Def.2.2) is associated with any subset $B_k \in \mathcal{A}$. Its requested rate $\mathcal{R}(B_k)$ is its sum-rate. Worst and best receivers (Def.3.2), resp. $\mathcal{O}_-$ and $\mathcal{O}_+$ are also illustrated.

The capacity region of the classical broadcast channel (BC) or multiple access channel (MAC) under memoryless Gaussian channel assumptions is well known for a predetermined set of users with fixed channels [1]. But the capacity region of a wireless cell where the users are defined by a spatial density is not yet defined. Recent results exploiting stochastic geometry obtained good estimates of the SINR distribution [2], [3] in cellular networks with randomly placed users. However, to the best of our knowledge, all rate expressions obtained from these distributions rely on pure time sharing strategies, thus underestimating the cell's capacity region [2]–[4]. In [5], the cell load is computed from an approximation of the cell size distribution. But this work estimates the requested sum-rate per cell (the throughput demand), and not the radio access capacity. The capacity metrics proposed in [4] and in [6] are interesting but do not correspond to the Shannon fundamental limit since time-sharing is again implicitely assumed. The gap between these metrics and a fundamental limit is not known. In [7], the fundamental limits are studied considering superposition coding, but the limit is given as the solution of an optimization problem. In [8] the EE-SE limit in a dense typical cell is evaluated for different tranmission schemes but the fundamental limit is not established.

A. Contributions of this paper

This paper follows [8] and proves that the EE-SE limit achieved by an ∞-user superposition coding is the fundamental limit that determines the access capacity regions of the SCBC and the SCMAC. The main results follow:

- The access capacity regions of the SCBC and SCMAC are defined and computed for Gaussian memoryless channels.
- The uniform capacities (symmetric rates) of the SCBC and SCMAC are established.
- The interest of the method is illustrated with a simple example.

Due to space constraints, some detailed proofs are only sketched briefly but are available in [9].

II. Model and notations

This paper is restricted to the study of Gaussian channels where the BS and nodes are equipped with single antennas. For the sake of clarity, the maximum rate simultaneously achievable by all nodes (the symmetric rate) [10] is used to illustrate the results. To avoid confusion between symmetry of channels and rates, we will rather refer to this assumption as the *uniform rate* assumption. The uniform capacity is then defined and computed. More broadly, the access capacity region is defined as the set of achievable rate spatial densities. What achievability means for a density of nodes is not straightforward and will be detailed in the following sections.

The SCBC is detailed in Section III and the SCMAC is studied in Section IV using the MAC/BC equivalence under a transferable power assumption [11] .

Consider a unique BS serving an area denoted by $\Omega \subset \mathbb{R}^2$ with a large number of nodes. We denote by $(\Omega, \mathcal{A}, m)$ the corresponding measurable space with $\mathcal{A}$ the Lebesgue $\sigma-$algebra and m the Lebesgue measure. Let x be a point in Ω.

Without lack of generality, the BS is assumed to be located at point $(0,0)$. Nodes appear randomly in continuous time and space on Ω. As such, they are not described by a discrete set but through a spatial density $u(x)$. For any subset $B \in \mathcal{A}$, the average number of nodes per time unit is given by

$$U(B) = \int_{x \in B} u(x) \cdot m(dx). \tag{1}$$

The average number of nodes associated with the whole cell area Ω is denoted by U_T.

***Definition** 2.1 (Requested rate density):* The requested rate density $\rho(x) : \Omega \to \mathbb{R}$ is a measurable function that represents the requested rate spatial density at point x.

- Note 1: $\rho(x)$ is normalized[1]
- Note 2: The requested rates are either in downlink (SCBC) or uplink (SCMAC) mode.

The cell spectral efficiency is called the sum-rate per channel use:

$$\eta_s = \int_{\Omega} \rho(x) \cdot m(dx). \tag{2}$$

The uniform rate condition is obtained when each node requests the same quantity of information denoted by $\mathcal{I}_0$, with $\rho(x) = \mathcal{I}_0 \cdot u(x)$. In this special case, we have $\eta_s = \mathcal{I}_0 \cdot U_T$.

The central question we are addressing is to find the maximal value of $\mathcal{I}_0$ (in case of uniform rate) achievable under some power constraint. This maximal value is termed the *uniform capacity*. By extension, we also assess the set of achievable spatial densities $\mathcal{D}_{\Omega}(P_m)$ called *the access capacity region*.

Defining the achievability of a rates spatial density is not straightforward. To this end, we use a spatial discretization of the continuum and we analyze its limit when the number of samples tends to ∞. The discretization is done by partitioning Ω into a collection of subspaces with the following rates:

***Property** 2.1:* The rate requested by a subset $B \in \mathcal{A}$ is given by

$$\mathcal{R}(B) \le \int_{B} \rho(x) \cdot m(dx), \tag{3}$$

with equality if all requested information streams are independent[2]. Independence is assumed in the rest of this paper.

***Definition** 2.2 (Physically feasible node):* For any subset $B \in \mathcal{A}$, a physically feasible (PF) node $v(B)$ is defined as a unique point $x \in B$ (either in Tx or Rx mode), having its own channel characteristics and which requests the sum-rate of B given by (3).

Then, for a partition $\mathcal{B} = \{B_k \in \mathcal{A}; k \in [1;K]\}$ of Ω, a set of PF nodes $\{v_1, \ldots, v_K\}$ can be selected. Therefore, the BS and this set of nodes form either a $K-$user BC or a $K-$user MAC, referred to as a PF network $\mathfrak{N}(\mathcal{B})$ and represents a discrete approximation of the continuum model. The accuracy of this approximation relies on the number of PF nodes, the larger, the better. Therefore, the approximation can be progressively refined thanks to a sequence of partitions, where the $(i+1)^{th}$ partition is built from the i^{th} partition by a splitting process.

***Definition** 2.3:* Consider a sequence of partitions $\mathcal{B}^{(i)}; i \in \mathbb{N}^+$ built from the singleton $\mathcal{B}^{(0)} = \{\Omega\}$ and a splitting process defined as $B_k^{(i)} \longrightarrow \left(B_{2k}^{(i+1)}, B_{2k+1}^{(i+1)}\right)$. A PF node $v_k^{(i)}$ is associated with each $B_k^{(i)}$ thus defining a sequence of *PF networks* $\mathfrak{N}(\mathcal{B}^{(i)})$ denoted by $\mathfrak{N}^{(i)}$ for short.
When $i \to \infty$, the size of $\mathfrak{N}^{(i)}$ grows (2^i nodes) but the requested rate per PF node tends to 0 while the network sum-rate remains constant.

III. Gaussian-SCBC: definition and properties

This section uses the former definitions to analyze the Gaussian-SCBC, or G-SCBC. The formal definition of the G-SCBC follows the classical G-BC definition with the difference that the output is a vector field over Ω instead of a

[1] $\rho(x)$ and related quantities ($\mathcal{I}_0$, $\mathcal{R}(\cdot)$, ...) are given in bits-per-channel-use (bpcu) throughout this paper.

[2] Independence, assumed throughout the rest of this paper, is a classical assumption in information theory.

discrete set of output values. Let Ξ be the set of piecewise integrable continuous functions $\Xi = \left\{\Phi(x) \in L_0 : \Omega \rightarrow \mathbb{R}^d\right\}$ where $\Phi(x)$ is a $d-$dimensional vector field on Ω.

***Definition** 3.1 (Gaussian-SCBC):*
Given:

- Ω, a subspace on a Hilbert space of dimension 2,
- $x_0 = (0, 0) \in \Omega$, the source,
- $\mathcal{Y}_c$, the coding alphabet used by the source to transmit a symbol $y \in \mathcal{Y}_c$,
- $\Xi = \left\{\Phi(x) : \Omega \rightarrow \mathbb{R}^d\right\}$ a set of fields on Ω,

the SCBC is a function that maps any input code y to a set of conditional probability density functions (pdfs) on Ξ:

$$\mathcal{H} : \left\{\mathcal{P}_{\Phi(.)|y}; \forall y \in \mathcal{Y}_c\right\}, \tag{4}$$

The set of conditional pdfs associated with the Gaussian-SCBC is

$$\mathcal{P}_{\Phi(x)|y} = \mathcal{N}\left(y, \nu(x)\right), \tag{5}$$

where $\mathcal{N}(\mu, \nu)$ denotes the normal distribution with mean μ and variance ν. In (5) the channel gain is normalized for each receiver which makes the equivalent noise variance proportional to the inverse of the channel gain $\nu(x) = \sigma^2/|h(x)|^2$, where σ^2 stands for the power of the assumed additive white Gaussian noise (AWGN).

A. Physically feasible (PF) networks

The capacity region of the G-SCBC is obtained thanks to the discretization process described in Section II.

A PF network $\mathfrak{N}$ is a BC with a source and a discrete set of receiver nodes. According to [12], a coding scheme in such a channel assigns a codeword of length n, $y^n = [y_0, \ldots, y_{n-1}]$ as a function of the messages $(m_1, \ldots, m_K)$ to be transmitted to the K receivers and thus generates n fields $\Phi^n(x)$.

***Definition** 3.2:* A *PF receiver* associated with a subset B_k is defined by two successive operations (observation and decoding) applied to the n fields restricted to B_k, denoted by $\Phi^n_{B_k}$:

$$\Phi^n_{B_k} \xrightarrow{\mathcal{O}} v^n_k \xrightarrow{\mathcal{D}} \hat{m}_k. \tag{6}$$

The observation operator $\mathcal{O}$ plays a fundamental role in the proposed analytical approach: it extracts a point information from the local field $\Phi^n_{B_k}$. Then, the decoder conventionally maps this observation to an estimate $\hat{m}_k$. While the observer is used to tune the discrete model, the decoder is usually chosen with the encoder to minimize the error probability.

We restrict a PF receiver to be able to observe a unique point x on B_k with two extreme cases: the best observer $\mathcal{O}_+$ selects the least noisy sample over B_k, and the worst observer $\mathcal{O}_-$ selects the noisiest sample (see Fig.1). For a partition $\mathcal{B}$, using either the best or the worst observers provide two discrete approximations of the SCBC, referred to as resp. the best or the worst PF networks, denoted by $\mathfrak{N}_+(\mathcal{B})$ and $\mathfrak{N}_-(\mathcal{B})$.

Both $\mathfrak{N}_+(\mathcal{B})$ and $\mathfrak{N}_-(\mathcal{B})$ are classical BCs and their capacity regions denoted by $\mathcal{C}(\mathfrak{N}_+)$ and $\mathcal{C}(\mathfrak{N}_-)$ are known [12]. Obviously, $\mathcal{C}(\mathfrak{N}_-) \subset \mathcal{C}(\mathfrak{N}_+)$.

***Definition** 3.3 (Relative achievability):* a rate spatial density $\rho(x)$ is achievable with respect to (w.r.t.) $\mathfrak{N}_\pm(\mathcal{B})$ if and only if $(\mathcal{R}(B_1), \ldots, \mathcal{R}(B_K)) \in \mathcal{C}(\mathfrak{N}_\pm)$ (where $\pm \in \{-, +\}$ indicates either the worst or the best PF network).

B. Access capacity region: definition

Def. 3.3 (relative achievability), and Def. 2.3 (sequence of partitions) are now used to established the following theorems. Consider a sequence of PF networks using the worst (resp. the best) receivers:

***Theorem** 3.1:* If a rate spatial density $\rho(x)$ is achievable w.r.t. $\mathfrak{N}_-^{(i)}$ for some $i \geq 0$, then $\rho(x)$ is achievable w.r.t. $\mathfrak{N}_-^{(j)}$, $\forall j \geq i$.

***Theorem** 3.2:* If a rate spatial density $\rho(x)$ is not achievable w.r.t. $\mathfrak{N}_+^{(i)}$ for some $i \geq 0$, then $\rho(x)$ is not achievable w.r.t. $\mathfrak{N}_+^{(j)}$, $\forall j \geq i$.

Proof: The proofs are provided in [9] and exploit the Markov chain rule. ■

The definition of the asymptotic achievability and the access capacity region then follow:

***Definition** 3.4:* A density $\rho(x)$ is said to be asymptotically achievable if there exists a sequence of partitions on Ω and a positive integer i_0 such that $(\mathcal{R}(B_1^{(i)}), \ldots \mathcal{R}(B_{K_i}^{(i)})) \in \mathcal{C}(\mathfrak{N}_-^{(i)})$ for all $i \geq i_0$.

***Definition** 3.5:* The access capacity region, denoted by $\mathcal{D}_\Omega(P_m)$, is the set of asymptotically achievable densities $\rho(x)$ for a given maximal transmission power P_m.

Th. 3.1 shows that if a partition $\mathcal{B}$ on Ω exists such that $\rho(x)$ is achievable w.r.t. $\mathfrak{N}_-(\mathcal{B})$, then $\rho(x)$ is asymptotically achievable. Alternatively, Th. 3.2 indicates that if $\rho(x)$ is not achievable w.r.t. $\mathfrak{N}_+(\mathcal{B})$, then $\rho(x)$ is not asymptotically achievable. Therefore, any partition $\mathcal{B}$ gives upper and lower bounds of the access capacity region. Clearly, these bounds are satisfied:

$$\mathcal{C}\left(\mathfrak{N}_-^{(0)}\right) \subset \mathcal{C}\left(\mathfrak{N}_-^{(1)}\right) \cdots \subset \mathcal{C}\left(\mathfrak{N}_-^{(\infty)}\right) \tag{7}$$

$$\mathcal{C}\left(\mathfrak{N}_+^{(0)}\right) \supset \mathcal{C}\left(\mathfrak{N}_+^{(1)}\right) \cdots \supset \mathcal{C}\left(\mathfrak{N}_+^{(\infty)}\right). \tag{8}$$

If the sequence of partitions is chosen such that the capacity regions with the worst and best receivers converge asymptotically to the same limit when $i \rightarrow \infty$ then this limit is the G-SCBC access capacity region. Worst PF networks provide successive achievable regions while best PF networks provide successive upper bounds. The limit is thus a capacity region in Shannon's sense (with achievability and converse).

C. Uniform capacity

For the sake of simplicity, it is useful to define the uniform capacity, which is a special solution of the limit of the access capacity region:

***Definition** 3.6:* Given a density of nodes $u(x)$, the uniform capacity of a BS with power P_m is

$$\mathcal{C}_u(P) = \sup_{\mathcal{I}_0; \mathcal{I}_0 \cdot u(x) \in \mathcal{D}_\Omega(P_m)} (\mathcal{I}_0 \cdot U_T). \tag{9}$$

The dual formulation follows:

***Definition** 3.7:* Given $u(x)$ and $\mathcal{I}_0$, the minimal power ensuring asymptotic achievability is given by

$$\tilde{P}_m = \min_{P \in \mathbb{R}^+} P; \rho(x) = \mathcal{I}_0 \cdot u(x) \in \mathcal{D}_\Omega(P).$$

Given the complementary cumulative distribution function (ccdf) of the equivalent noise level:

$$G_\nu(\nu) = \frac{1}{U_T} \cdot \int_\Omega u(x) \cdot \mathbb{1}\left[\nu(x) \geq \nu\right] \cdot m(dx), \qquad (10)$$

and the pdf $f_\nu(\nu) = -G'_\nu(\nu)$, we have:

***Theorem** 3.3:* The minimal power required to serve a node spatial density $u(x)$ with a uniform rate $\mathcal{I}_0$, is

$$\tilde{P}_m = a \cdot \eta_s \int_{\nu_m}^{\nu_M} t \cdot f_\nu(t) \cdot e^{a \cdot \eta_s \cdot G_\nu(t)} \cdot dt, \qquad (11)$$

where $a = 2\log(2)$.

Proof: The complete proof is available in [9]. A specific sequence of partitions is built from $B_0^{(0)} = \Omega$, and by splitting recursively each subset $B_k^{(i)}$ into two subsets such that $B_{2k}^{(i+1)} = \left\{x \in B_k^{(i)}; \nu(x) < \bar{\nu}_k^{(i)}\right\}$ and $B_{2k+1}^{(i+1)} = \left\{x \in B_k^{(i)}; \nu(x) \geq \bar{\nu}_k^{(i)}\right\}$ where $\bar{\nu}_k^{(i)}$ is some threshold value. In such partition the subsets are ordered w.r.t. their noise levels, i.e. $\forall k, q > k, x \in B_k^{(i)}, x' \in B_q^{(i)}$, then $\nu(x) \leq \nu(x')$.

Such a partition is illustrated in Fig. 2. At a given iteration i, a superposition coding algorithm (for either the worst or the best PF network) that minimizes the sum-power, allocates the following powers from $k = 0$ to $K(i)$:

$$\mathfrak{N}_\pm : P_{\pm,k}^{(i)} = \left(2^{2\mathcal{R}_k^{(i)}} - 1\right) \cdot \left(\nu_{\pm,k}^{(i)} + \sum_{q<k} P_{\pm,q}^{(i)}\right), \qquad (12)$$

with $\nu_{\pm,k}^{(i)}$ the highest (resp. lowest) noise level of the k^{th} subset. The sum-power is then given by $\Pi_{\pm,k}^{(i)} = \sum_{q \leq k} P_{\pm,q}^{(i)}$. According to Theorems 3.1 and 3.2, this expression provides either an upper bound (with $\mathfrak{N}_-$) or a lower bound (with $\mathfrak{N}_+$) on $\tilde{P}_m$. Further, they converge to the same value when $i \to \infty$.

Considering that $\lim_{i \to \infty} \left(2^{2\mathcal{R}_k^{(i)}} - 1\right) \approx 2\mathcal{R}_k^{(i)}$, the sum-power derives from (12) using finite differences equivalence:

$$\dot{\Pi}(\nu) = a \cdot \eta_s f_\nu(\nu) \cdot (\nu + \Pi(\nu)), \qquad (13)$$

whose solution provides the expression in Th. 3.3. ■

D. Access capacity region: results

The former result can be now extrapolated to obtain the access capacity region (Def. 3.5). Without loss of generality, we may write $\rho(x) = \eta_s \cdot f_\rho(x)$ where $f_\rho(x)$ is the normalized rate spatial density. Then, the access capacity region relies on finding the maximal value of η_s for any distribution $f_\rho(x)$ such that $\rho(x) \in \mathcal{D}_\Omega$. Theorem 3.3 still applies with (10) replaced by

$$G_\nu(\nu) = \int_\Omega f_\rho(x) \cdot \mathbb{1}\left[\nu(x) \geq \nu\right] \cdot m(dx). \qquad (14)$$

Theorem 3.3 is of great interest since it characterizes the set of achievable rate spatial densities. The result is expressed as a function of $f_\nu(\nu)$ and (14) allows us to map any rate spatial density $\rho(x)$ into a noise distribution $f\nu(\nu)$.

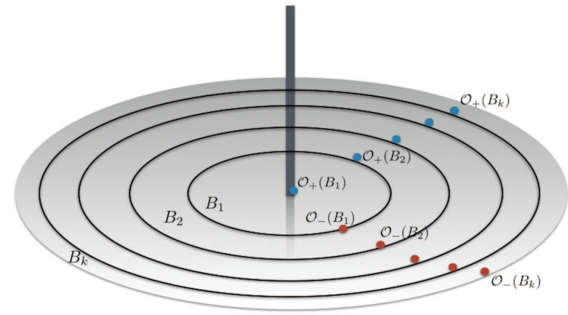

Fig. 2. Illustration of the partition of Ω in homogeneous regions, with the corresponding best (blue) and worst (red) receivers.

IV. GAUSSIAN-SCMAC: DEFINITION AND PROPERTIES

We now consider the uplink scenario modeled as an SC-MAC with transferable powers.

***Definition** 4.1 (Gaussian-SCMAC):*
Given the following:

- Ω, a subspace on a Hilbert space of dimension 2,
- $\mathcal{Y}_c$, a coding alphabet,
- $\Xi = \{\Phi(x) : \Omega \to \mathcal{X}_c\}$, a set of input fields on Ω,
- $x_0 = (0, 0) \in \Omega$, the receiver position,
- $z \in \mathbb{R}^d$ the channel output (observed at the BS).

The G-SCMAC is a function that maps any input field $\Phi(x)$ to a set of conditional pdfs on $\mathbb{R}^d$:

$$\mathcal{H} : \left\{\mathcal{P}_{z|\Phi(.)}; \forall \Phi(.) \in \Xi\right\}, \qquad (15)$$

with

$$\mathcal{P}_{z|\Phi(x)} = \mathcal{N}\left(\int_\Omega h(x) \cdot \Phi(x) dx, \sigma^2\right). \qquad (16)$$

The relation between the G-SCMAC definition and the usual MAC is less immediate than in the case of the G-SCBC. The input is the spatial source field $\Phi(x)$ and the output is the observation at the output $(0, 0)$. The continuum will be now approximated through a discretization process that selects a discrete set of point sources.

A. Physically feasible (PF) networks

***Definition** 4.2:* A *PF transmitter* associated with a subset B_k is defined by an encoder that maps a message to a sequence of symbols and a selector $\mathcal{S}$ that selects one transmission point $x_k \in B_k$:

$$m_k \xrightarrow{\mathcal{C}} y_k^n \xrightarrow{\mathcal{S}} \Phi_{B_k}^n = y_k^n \cdot \delta(x_k), \qquad (17)$$

with $\delta(x)$ the delta function on Ω.

The input field in B_k is controlled by putting a coded point source on x_k. The best selector, denoted by $\mathcal{S}_+$, selects the point with the best (i.e., least) pathloss, while the worst selector, denoted by $\mathcal{S}_-$, selects the point with the worst pathloss. The best and the worst PF networks associated with a partition $\mathcal{B}$ are again denoted by $\mathfrak{N}_+(\mathcal{B})$ and $\mathfrak{N}_-(\mathcal{B})$.

Under these assumptions and for a given selector, $\mathfrak{N}(\mathcal{B})$ is equivalent to a classical G-MAC. Indeed, one may write the BS signal as

$$z(n) = \sum_k h(x_k) \cdot y_k(n) + \eta(n). \tag{18}$$

B. Access capacity region

Thanks to the duality principle between the BC and MAC [11], the access capacity region of the SCMAC with transferable power is strictly equal to that of the dual SCBC[3].

For the G-SCMAC, the dual algorithm of superposition coding is successive interference cancellation (SIC) where appropriate individual powers are used and the decoding order is from the best to the worst channel user. An additional question then concerns the power allocation policy: since transferable powers are considered, what is the optimal power used by each node?

***Theorem** 4.1:* Given $\rho(x)$, a requested rate spatial density of a G-SCMAC. The minimum sum-power is obtained when the BS successively decodes the streams from the nearest to the furthest node. The corresponding optimal individual transmission power per node $\tilde{P}_t$ is given by

$$P_t(x) = \nu(x) \cdot \rho(x) \cdot e^{a\eta_s \cdot G_\nu(\nu(x))}. \tag{19}$$

Proof: A straightforward proof relies on selecting arbitrarily these individual powers and to show that an SIC receiver is capacity achieving. Thanks to MAC-BC duality [11], Th. 3.3 gives the minimal sum-power.

A more complete proof would rely on computing the recursive powers with the G-SCMAC as detailed in [9]. ■

V. Application example

A basic scenario made of an isolated circular cell $(0, R)$ is considered in either downlink or uplink. A simple power-law pathloss model and an omnidirectional antenna are considered with no shadowing: $|h(x)|^2 = h_0 \cdot |x|^{-\alpha}$, where h_0 and α represent resp. the reference pathloss and the attenuation slope. The focus is put on the uniform capacity, i.e. with $u(x) = u_0$ over Ω and with a constant information per packet $\mathcal{I}_0$ (so that $\eta_s = \mathcal{I}_0 \cdot U_T$).

Under these assumptions, f_ν is given by

$$f_\nu(\nu) = \frac{2}{\alpha} \cdot \left(\frac{\nu}{\nu_R}\right)^{2/\alpha - 1}, \tag{20}$$

where ν_R stands for the equivalent noise at the cell edge.

Using (20) in (10) leads to

$$\tilde{P}_m = \nu(R) \cdot a \cdot \eta_s \cdot e^{a \cdot \eta_s} \cdot \Gamma\left(1 + \frac{\alpha}{2}, a \cdot \eta_s\right), \tag{21}$$

with $\Gamma(a, x)$ the incomplete gamma function. Eq. (21) provides the fundamental EE-SE limit of this cell. Given the power normalized by the equivalent noise at the cell edge

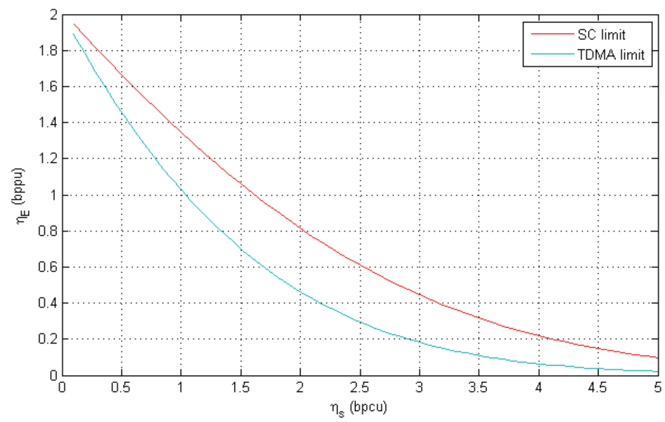

Fig. 3. The EE-SE fundamental limit of a circular cell, obtained by the continuum extension of the superposition coding principle (SC) and the EE-SE limit with pure TDMA.

$(p_r = P_M/\nu(R))$, and the EE $\eta_E = \eta_S/p_r$, expressed in bits per power use (bppu)[4],the EE-SE fundamental limit is represented in Fig. 3. For comparison purpose, the EE-SE achievable region with pure time-division multiple access (TDMA) is also represented. The TDMA limit is obtained with a joint power-bandwidth optimization as described in [8]. The superposition coding (SC) upperbound clearly outperforms TDMA. The relative EE gain is increasing with SE.

By inverting (21), the cell uniform capacity can be expressed in a way similar to that of the single user Shannon capacity:

$$\eta_s \leq \mathcal{C}_0(P_m) = a^{-1} \cdot C_{U,\alpha}(\gamma_R), \tag{22}$$

where γ_R is the SNR at the cell edge and $C_{U,\alpha}(\cdot)$ is the inverse function of $f(x) = x^{2+\frac{\alpha}{2}} \cdot {}_1F_1\left(1; 2 + \frac{\alpha}{2}; x\right)$ with ${}_1F_1(a; b; x)$ the confluent hypergeometric function (Sec. 9.21; [13]).

Interestingly, this uniform capacity relies only on the channel power law, the node density, and the cell edge SNR. This capacity is plotted in Fig. 4 (dark blue) with the TDMA achievable rate (dark green). From a practical perspective, achieving the theoretical limit would rely on using SC with an infinite number of nodes and an infinite coding length (i.e. a doubly asymptotic regime). However, the limit can be approached by splitting the service area into K subsets as a function of noise levels. Then the BS should use a simple time-sharing inside each group and SC between them. Dash-dot curves in Fig. 4 represent achievable rates (i.e. with worst receivers) with $2^n, n \in \{2, 3, 5\}$ subsets while dashed curves represent successive converse bounds (with best receivers).

Lastly, we also provide in Fig. 5 the optimal transmission power in uplink mode (MAC) to be allocated per node as a function of the node-BS distance, according to (19). Surprisingly, the maximal power is not obtained for the furthest nodes. Additionally, the nearest nodes receive also energy gain. These

[3]It is worth mentioning that the SCMAC capacity with transferable powers can also be computed following a procedure similar to that of Th. 3.3, which is not presented here due to lack of place. The MAC/BC duality is used for each PF network, and this is enough to obtain the G-SCMAC capacity.

[4]$\nu(R)$ is a system parameter that represents the product of the AWGN variance and the inverse of the worst pathloss in the cell, i.e. the maximal equivalent noise. It has a dimension of energy. p_r may be interpreted as a gain w.r.t. to $\nu(R)$.

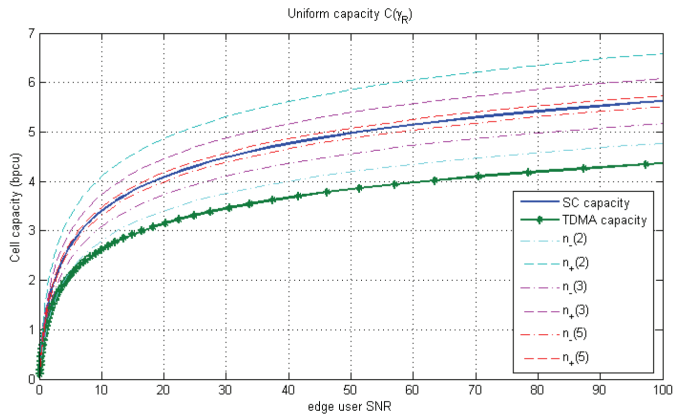

Fig. 4. Uniform capacity for a cell ($\alpha = 3.65$). The fundamental limit (SC) and TDMA achievable rate (TDMA) are plotted. Successive approximations with either best (n_+) or worst (n_-) receivers, under a partition of 2^n subsets (4, 8 or 32) are drawn.

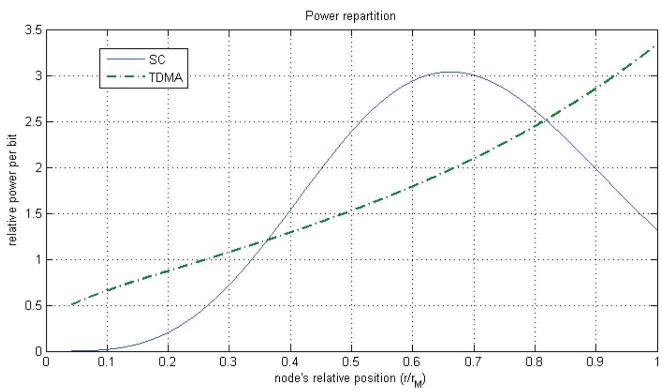

Fig. 5. Relative power transmission levels as functions of the distance between the BS and the transmitting node to achieve optimal transmission for $\alpha = 3.65, \eta_S = 3$ with a sub-optimal TDMA or an ideal SC.

results are due to the SC principle. It could be interesting to search for a decentralized optimal power allocation to minimize the maximal individual power. This is left for future work.

VI. Conclusion and Perspectives

In this paper we have proposed definitions of the G-SCBC and the G-SCMAC representing a wireless cell in downlink or uplink modes. We have defined the uniform capacity and the access capacity region and have given general expressions for them. These expressions have been established with the definition of an appropriate sequence of partitions on Ω, which gave successively refined approximations of the spatial continuum model. The access capacity region has been established at the limit of this sequence.

This access capacity region has an important physical meaning. A cell in which users arrive randomly (but under a stationnary process), with individual rate requests, is characterized by its average rate spatial density $\rho(x)$. The minimal power required to serve these users has been established. To achieve this fundamental limit however, the BS would need to know in advance all the rate requests to superpose all transmissions optimally. This is not realistic since delay constraints are usually present. However, our work provides a fundamental limit that may be used to evaluate the relative efficiency of any scheduler w.r.t. this delay tolerant bound.

This study has been limited to Gaussian channels, and further work is planned to broaden this model to more complex scenarios. A first step will be to consider fading channels with or without channel state information using the known capacity results for the parallel BC [14]. Multiple-antenna (MIMO) channels need also to be considered and the fact that we are dealing with doubly asymptotic scenarios may simplify such a study. Last but not least, multi-BS scenarios are of interest either in massive MIMO or interferering multicells modes.

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